# Effect of non-axisymmetric $\mathbf{B}_0$ on modes in cores

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## Main points

- Traditionally, background magnetic fields $\mathbf{B}_0$ are simple, e.g. spatial uniformity, axisymmetry, or perfect boundary conditions have been used for mathematical simplification
- Differences in torsional Alfvén modes between axisymmetric and non-axisymmetric $\mathbf{B}_0$ are compared
- The non-axisymmetry leads to larger scale mode filling the volume of the core

## Motivation

- The magnetic field within the Earth's liquid outer core is inaccessible through direct downward projection of the magnetic field observed above Earth's surface.
- Previously, torsional Alfvén waves have been used to constrain the cylindrical radial component of the steady magnetic field deep in the core (Gillet et al., 2010). Recently, non-zonal columnar waves have been identified in satellite geomagnetic data that could extend the constrains given by torsional Alfvén waves (Gillet et al., 2022).
- To link the propagation of any of these waves to the steady large-scale magnetic field in the core, it is crucial to understand the influence of different magnetic field morphologies on the periods, velocities and magnetic field perturbations associated to the waves.

## Axisymmetric $\mathbf{B}_0$

- When using an axisymmetric $\mathbf{B}_0$, the system is decoupled in the azimuthal degree $m$.
- We revisit some of the results presented by Luo and Jackson (2022).
- Dipolar field $\mathbf{B}_{0,1} \sim \nabla\times\nabla\times(5r-3r^3)Y_1^0\mathbf{r}$
- Quadrupolar field $\mathbf{B}_{0,2} \sim \nabla\times\nabla\times f(r)Y_2^0\mathbf{r}$
- $\mathrm{Le}=10^{-4}$, $\mathrm{Lu}=2/\mathrm{Le}$
- Truncation: maximum SH degree $L=200$, radial degree $N=(L-l)/2$.

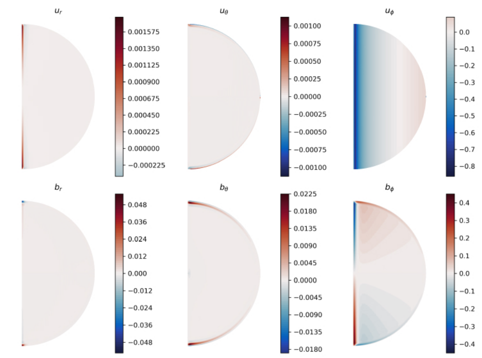


**Figure 1:** Gravest TM in a dipolar field $\mathbf{B}_{0,1}$. $Q=49.3$, $\lambda=-0.013530748466+0.666958322798i$

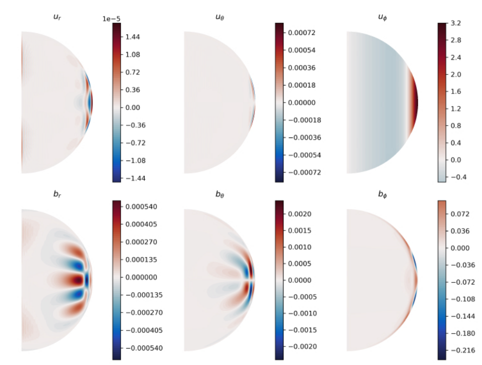


**Figure 2:** Gravest TM in a quadrupolar field $\mathbf{B}_{0,2}$. $Q=156.7$, $\lambda=-0.006595246178+1.033595994289i$

## Linear mode calculation

The inviscid MHD equations are written as

$$\frac{\partial\mathbf{u}}{\partial t}=-\frac{2}{\mathrm{Le}}\mathbf{1}_z\times\mathbf{u}-\nabla p+\nabla\times\mathbf{B}\times\mathbf{B}_0+\nabla\times\mathbf{B}_0\times\mathbf{B},$$
$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times\mathbf{u}\times\mathbf{B}_0+\frac{1}{\mathrm{Lu}}\nabla^2\mathbf{B},$$

with the Lehnert number $\mathrm{Le}=B_0/(R\Omega\sqrt{\mu\rho})$ and Lundquist number $\mathrm{Lu}=B_0R/(\eta\sqrt{\mu\rho})$.

The velocity and magnetic field are represented by a 3D poloidal-toroidal basis, satisfying the regularity at the origin and the appropriate boundary condition at the surface. Using a Galerkin projection of the bases $\mathbf{u}_i$ and $\mathbf{B}_i$ onto the momentum and induction equation, gives

$$\int\mathbf{u}_i\cdot\left(\lambda\mathbf{u}_j+\frac{2}{\mathrm{Le}}\mathbf{1}_z\times\mathbf{u}_j-\nabla\times\mathbf{B}_j\times\mathbf{B}_0-\nabla\times\mathbf{B}_0\times\mathbf{B}_j\right)\mathrm{d}V=\mathbf{0},$$
$$\int\mathbf{B}_i\cdot\left(\lambda\mathbf{B}_j-\nabla\times(\mathbf{u}_j\times\mathbf{B}_0)-\mathrm{Lu}^{-1}\nabla^2\mathbf{B}_j\right)\mathrm{d}V=\mathbf{0},$$

Solutions are the

- slightly modified inertial modes (IM),
- torsional Alfvén modes (TM),
- and Magneto-Coriolis modes (MCM).

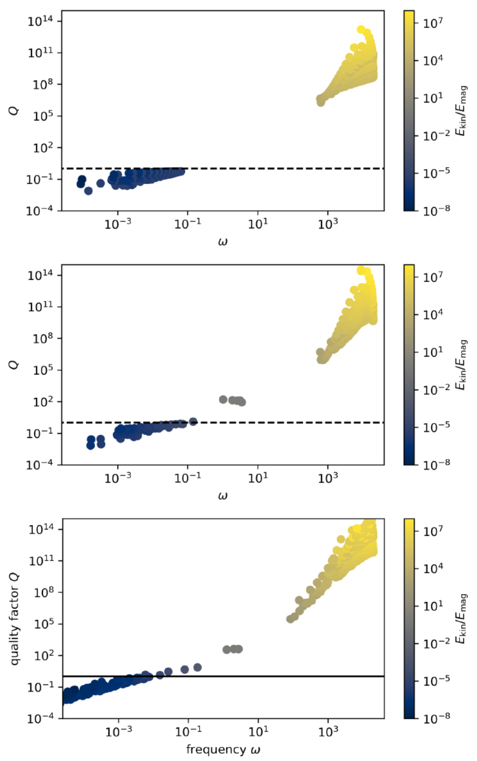


**Figure 3:** Frequency-Quality factor spectrum of *converged* modes for $\mathbf{B}_{0,1}$ ($L=50$, top), $\mathbf{B}_{0,2}$ ($L=50$, middle), $\mathbf{B}_{0,3}$ ($L=17$, bottom).

## Non-axisymmetric $\mathbf{B}_0$

- $\mathbf{B}_{0,3}\sim\nabla\times\nabla\times(f_1(r)Y_1^0+f_2(r)Y_1^1)\mathbf{r}$

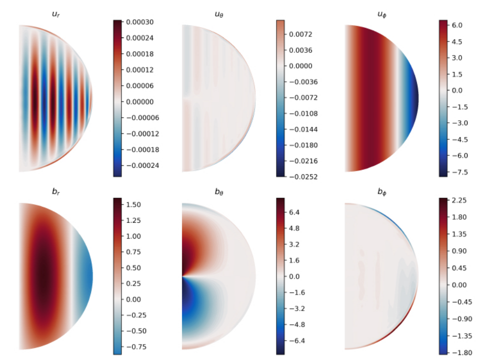


**Figure 4:** Gravest TM in a non-axisymmetric field $\mathbf{B}_{0,3}$. $Q=153.5$, $\lambda=-0.008268831753+1.269330959934i$

- Truncation is $L=100$, $N=(L-l)/2$, $m\leq l$

## Convergence

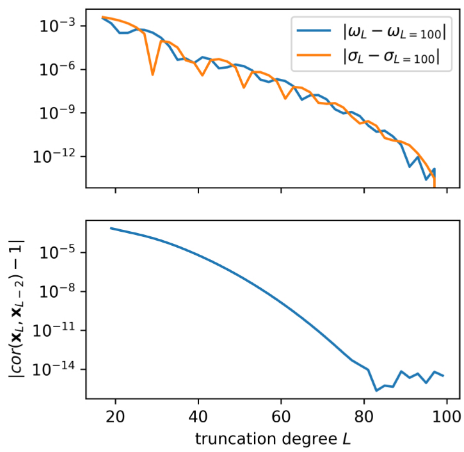


**Figure 5:** Convergence of gravest TM in a non-axisymmetric field

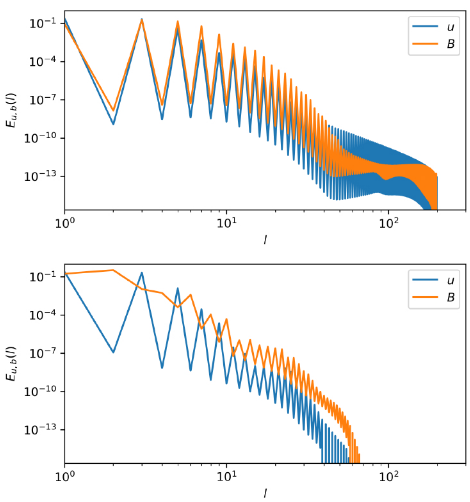


**Figure 6:** SH degree spectra for quadrupolar field $\mathbf{B}_{0,2}$, $L=200$ (top) and non-axisymmetric field $\mathbf{B}_{0,3}$, $L=100$ (bottom)

- Despite the need for all azimuthal orders $m$, gravest TM converges more quickly
- Smaller max. SH degree needed to achieve convergence

## 1D Torsional Alfvén mode equation

The diffusionless one dimensional TM equation reads

$$s^3H\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial s}\left(Hs^3\frac{\partial\xi}{\partial s}\left\langle B_{0,s}^2\right\rangle\right),$$

with $\xi=u_\phi(s)/s$, $H$ the column half height and

$$\left\langle B_{0,s}^2\right\rangle=\frac{1}{4\pi sH\mu_0\rho}\oint\int_{-H}^{H}B_{0,s}^2 s\,\mathrm{d}z\,\mathrm{d}\phi.$$

Luo and Jackson (2022) derive an approximate 1D set of equations including magnetic diffusion, that introduces a dependency on $\langle B_sB_\phi\rangle$.

## Discussion & Outlook

- A non-axisymmetric magnetic field increases scale in spatial structure of TM.
- This is understood with $\langle B_s^2\rangle$ dependency in the 1D TM equation.
- Effect of $\mathbf{B}_0$ on MCM more intricate and is part of ongoing investigations.
- Can a collection of modes provide constraints on the steady background field of the Earth?

## References

N. Gillet, D. Jault, E. Canet, and A. Fournier. *Nature*, 465(7294):74–77, 2010.
N. Gillet, F. Gerick, D. Jault, T. Schwaiger, J. Aubert, and M. Istas. *PNAS*, 119(13):e2115258119, 2022.
J. Luo and A. Jackson. *Proc. R. Soc. A*, 478(2259):20210982, 2022.