Multifidelity Orbit Uncertainty Propagation in the Presence of Stochastic Accelerations

Alberto Fossà[1] Daniel Alazard[1] Emmanuel Delande[2] Roberto Armellin[3]

[1]Institut Supérieur de l'Aéronautique et de l'Espace, Toulouse, France

[2]Centre National d'Études Spatiales, Toulouse, France

[3]The University of Auckland, Auckland, New Zealand

Context

Accurate propagation of uncertainties is key in space situational awareness (SSA). Ranging from Space surveillance and tracking (SST) to collision avoidance maneuver (CAM) planning, several activities require the estimation of the space object (SO) state at a future epoch given the initial conditions (ICs) at an earlier time. Orbit determination (OD) procedures return only a probabilistic description of the initial SO state. Since an analytical solution to the uncertainty propagation (UP) problem does not exist in the nonlinear orbital dynamics, an approximation of the transformed probability density function (pdf) must be computed. Accurate and computationally efficient algorithms for orbit UP are hence sought to answer the needs of space surveillance operators facing a continuously growing space population.

Mathematical Background

- **Differential algebra (DA)**, a computing technique based on the idea that is possible to extract more information from a function f that its mere value $y = f(x)$ at x [1]. It allows to compute the Taylor expansion of the flow of differential equations in terms of their ICs.
- **Automatic domain splitting (ADS)**, an algorithm to control the error committed by approximating the transformation with a Taylor polynomial [2]. If the initial uncertainty is large, several polynomials are needed to retain the accuracy of the estimate.
- **Gaussian mixture model (GMM)**, a tool to describe arbitrarily shaped pdfs as a finite sum of Gaussian kernels [3]. Gaussianity is not preserved in the nonlinear orbital dynamics.
- **Multifidelity (MF) approach**, a technique that combines low-fidelity (LF) and high-fidelity (HF) dynamical models to obtain an accurate estimate at a lower computational cost [4].

Low-order automatic domain splitting (LOADS)

A novel splitting algorithm tailored to second order Taylor expansions was developed by the authors and employed in this work [5]. A DA-based nonlinearity index (NLI) was proposed and used to detect the needs for splitting. Given the nonlinear transformation $[\boldsymbol{y}] = \boldsymbol{f}([\boldsymbol{x}])$ evaluated in DA, the NLI is defined as

$$\nu = \frac{\|\boldsymbol{B}\|_2}{\left\|\bar{\boldsymbol{J}}\right\|_2}$$

with $[\boldsymbol{J}] = \partial \boldsymbol{f} / \partial \boldsymbol{x}$ Jacobian of the transformation, $= \bar{\boldsymbol{J}}$ constant part of $[\boldsymbol{J}]$, $\boldsymbol{B}$ polynomial bounds of the nilpotent part of $[\boldsymbol{J}]$ and $\|\cdot\|_2$ Frobenius norm.

Application to orbit UP

When applied to orbit UP, the LOADS scheme results in the following recursive algorithm, for which an example output is shown in fig. 1.

1. **Initialization**. The orbit state is initialized as a DA vector $[\boldsymbol{x}_0] = \bar{\boldsymbol{x}}_0 + \boldsymbol{\beta} \cdot \delta\boldsymbol{x}$ with $\boldsymbol{x}_0$ nominal state, $\delta\boldsymbol{x}$ first order variations and $\boldsymbol{\beta}$ scaling parameters.
2. **Iterative propagation**. $[\boldsymbol{x}_0]$ is propagated until $\nu < \varepsilon_\nu \wedge t < t_f$ with ε_ν nonlinearity threshold and t_f target epoch.
   - If $t = t_f$, $[\boldsymbol{x}(t)]$ is added to the output set.
   - If $\nu > \varepsilon_\nu \wedge t < t_f$, $[\boldsymbol{x}(t)]$ is split into three polynomials and the propagation resumed on each new expansion.

   This procedure is iterated until all polynomials have reached t_f.
3. **Final solution**. The output is a set, or manifold, of polynomials $\{[\boldsymbol{x}^i(t_f)]\}$ at target epoch.

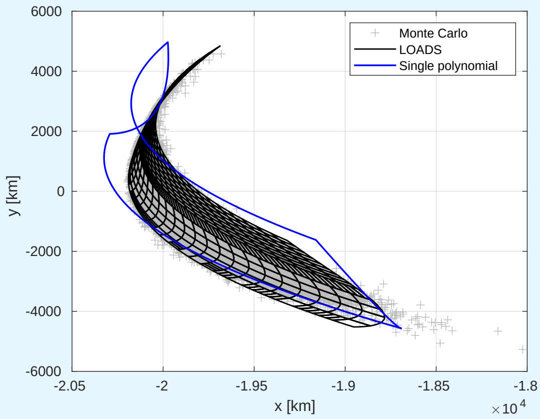

Figure 1. Aposelene passage after 2.5 revolutions for an orbit with $r_p = 6678\,\mathrm{km}$ and $e = 0.5$

Gaussian mixture model

GMM can approximate an arbitrary pdf as a weighted sum of Gaussian kernels thus providing an analytical description of the transformed state pdf

$$p(\boldsymbol{x}) \sim \sum_{i=1}^{N} \alpha_i p_g(\boldsymbol{x}; \boldsymbol{\mu}_i, \boldsymbol{P}_i)$$

An adaptive mixture model is build within the DA-LOADS framework by associating each polynomial $[\boldsymbol{x}^i(t)]$ to a kernel $p_g(\boldsymbol{x}; \boldsymbol{\mu}_i, \boldsymbol{P}_i)$ such that

- The optimal number of kernels N is automatically determined by the splitting procedure.
- Kernels means and covariances are propagated from t_0 to t_f by means of cheap polynomial evaluations on unscented transform (UT) sigma points.

Kernels are split along the spectral direction of maximum nonlinearity with a univariate splitting library that minimizes the L_2 distance between former and split pdfs [3].

Multifidelity orbit uncertainty propagation

Consider the SO's state at time t_0 described by a multivariate normal distribution with mean $\boldsymbol{\mu}_0$ and covariance $\boldsymbol{P}_0$. The workflow of the developed multifidelity UP algorithm is as follows.

1. **Initialization**. The orbit state is initialized as a DA vector as

$$[\boldsymbol{x}_0] = \boldsymbol{\mu}_0 + \boldsymbol{V} \cdot \left(c\sqrt{\boldsymbol{\lambda}}\delta\boldsymbol{x}\right)$$

   with $\boldsymbol{P}_0 = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^T$, $\boldsymbol{\lambda} = \mathrm{diag}(\boldsymbol{\Lambda})$, c confidence interval (CI) and $\delta\boldsymbol{x}$ first order variations.
2. **Computation of the expansion of the flow**. The Taylor expansion of the flow around the nominal ICs is computed with the LOADS algorithm using a LF dynamical model.

$$[\boldsymbol{x}_0] \to \left\{[\boldsymbol{x}_{LF}^i(t)]\right\}$$

3. **Propagation of the polynomial centers**. The centers of each polynomial at t_0 are propagated to the target epoch using an HF dynamical model.

$$\bar{\boldsymbol{x}}_0^i \to \bar{\boldsymbol{x}}_{HF}^i(t)$$

4. **Correction of the LF solution**. The polynomial expansions computed in LF are re-centered on the HF trajectories to restore the accuracy of the final solution.

$$[\boldsymbol{x}_{MF}^i(t)] = \bar{\boldsymbol{x}}_{HF}^i(t) + \left([\boldsymbol{x}_{LF}^i(t)] - \bar{\boldsymbol{x}}_{LF}^i(t)\right)$$

5. **Propagation of the kernels statistics**. The transformed pdf is estimated as

$$p(\boldsymbol{x}(t)) \sim \sum_{i=1}^{N} \alpha_i p_g(\boldsymbol{x}; \boldsymbol{\mu}_i(t), \boldsymbol{P}_i(t))$$

   where $\boldsymbol{\mu}_i(t), \boldsymbol{P}_i(t)$ are obtained evaluating $[\boldsymbol{x}_{MF}^i(t)]$ on UT sigma points.

Application to LEO regime

Figure 2 shows an application of the multifidelity UP algorithm in the low Earth orbit (LEO) regime. The LF dynamical model is the analytical Simplified General Perturbations (SGP4). The HF one is the perturbed Keplerian motion including Earth spherical harmonics, third-bodies attractions, Solar radiation pressure (SRP) and atmospheric drag.

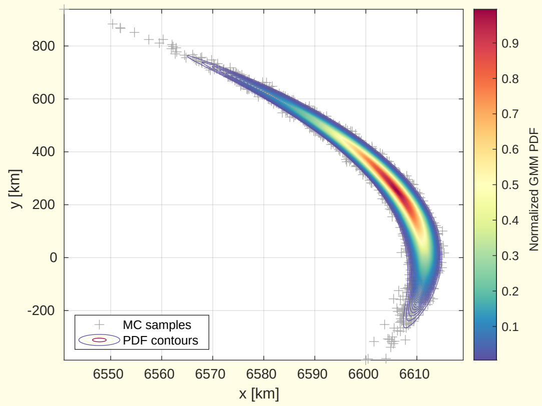

Figure 2. Propagated state pdf after two revolutions

Future Work

- Inclusion of process noise effects due to neglected or mismodeled perturbations. Two algorithms are under investigation for the propagation of white and colored noise respectively.
- Development of a Gaussian sum filter (GSF) for robust OD. Preliminary results were presented at the 73rd International Astronautical Congress (IAC) [6].

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 alberto.fossa@isae-supaero.fr